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TECHNICAL REPORT

THE INSTABILITY OF EX POST AND ROBUST AGGREGATION WITHOUT STATE–CONSEQUENCE SEPARATION

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Abstract

We prove the results of Hild (2001a, 2001b) in a framework that does not presuppose the separation of states and consequences. We introduce additional assumptions to make up for the structure that is lost by abandoning the representation of acts as functions from states to consequences.

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The Instability of Ex Post and Robust Aggregation without State–Consequence Separation

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1 Introduction

In Hild (2001a, 2001b), we have shown that both the ex post and robust mode of preference aggregation are not stable under refinements of the individuals' decision–theoretic models. The present report states and proves these results in a framework that does not presuppose the separation of states and consequences. We will use Fishburn's (1964, 1970) framework that contains only the minimal elements of any type of decision theory.[1] Fishburn starts with two primitives, consequences $C \in \mathcal{P}$ and acts $f \in \mathcal{F}$. Consequences are evaluated by a utility function; acts induce belief measures P_f over consequences. Acts, thus, occur only as indices of the belief measures which they induce and their underlying structure (e.g., as functions from worlds to consequences) is not considered.

Example Savage. In Savage's framework, Fishburn's probability measures P_f (for $f \in \mathcal{F}$) are induced by some common probability measure over a set Ω of 'states of the world'. Acts are identified with functions $f : \Omega \to \Gamma$ from states of the world to consequences. Hence, $P_f(c) := P(f^{-1}(c))$ for all $c \in \Gamma$.

The results of this report apply to the same utility aggregation rules and robust aggregation rules as our previous results but they cover only a smaller class of decision theories. The present results do not cover ordinal theories (e.g., decision–theoretic leximin or leximax) or any other theory based on binary evaluations. In theories of this type, we cannot infer nullness of a belief measure from preferences without making use of a state–consequence separation. More importantly, theories like Loomes/Sugden's (1982) regret theory presuppose Savage's representation of acts as functions from states to consequences and are thus incompatible with the present framework. The results of this report do, however, cover the following decision theories:

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[1] We note that, in Fishburn's framework, the separation of individual probabilities and utilities on the basis of preferences requires external randomization (roulette–lotteries; cf. Balch/Fishburn, 1974) or similar means.

- expected utility.
- expected utility with threshold (Fishburn, 1988).
- Choquet–expected utility (Gilboa, 1987, Schmeidler, 1989).
- probability transforms (Edwards, 1955, Kahneman/Tversky, 1979, Karmarkar, 1978).
- weighted utility theory (Chew 1983, Fishburn, 1983).
- Machina (1982).

The reader may consult the end of Section 2 for details.

2 Ex Post Aggregation

Generalized decision theory

Choose a frame of reference Γ for consequences and a frame of reference Φ for acts. We assume that Γ is at least countably infinite and that Φ has at least two elements. Throughout, we will only construct models with a finite number of consequences. We say that $\mathcal{P}$ is a Γ*–partition* if and only if $\mathcal{P}$ is a finite collection of non–empty and mutually disjoint sets the union of which is Γ. Acts will now induce belief measures for the consequences in the set $\mathcal{P}$. Instead of using belief measures on the power set of $\mathcal{P}$, we choose a more economical but equivalent approach and use belief measures on $[\mathcal{P}] := \{\bigcup X | X \subseteq \mathcal{P}\}$. The set $[\mathcal{P}]$ contains all events expressible in $\mathcal{P}$. As far as decision theory is concerned, we will assume that individual models use a monadic decision rule, are non–trivial, invariant under empty refinements and one–refinable in a sense to be defined. We will require any ex post social choice rule to evaluate consequences by aggregating individual utilities, to rank actions based on its evaluation of their consequences and to prefer absolutely dominant acts.

For some fixed $L \in \mathbb{N}^+$, we say that u is a *utility on* $\mathcal{P}$ if and only if $u : \mathcal{P} \to \mathbb{R}^L$. We admit multi-dimensional utilities in order to include models like those of Machina (1983) (cf. applications). Let $\mathbf{u}$ be the set of all u such that there exists some Γ–partition $\mathcal{P}$ on which u is a utility. For any Γ–partition $\mathcal{P}$, we define $\mathbf{u}(\mathcal{P})$ as the set of all utilities on $\mathcal{P}$. For $\langle a_l \rangle, \langle b_l \rangle \in \mathbb{R}^L$, we write $\langle a_l \rangle \geq \langle b_l \rangle$:iff $a_l \geq b_l$ for all $1 \leq l \leq L$; we write $\langle a_l \rangle = \langle b_l \rangle$:iff $a_l = b_l$ for all $1 \leq l \leq L$; we write $\langle a_l \rangle > \langle b_l \rangle$:iff $a_l > b_l$ for all $1 \leq l \leq L$.

For any Γ–partitions $\mathcal{P}$ and $\mathcal{P}'$ and any mapping $\phi : \mathcal{P} \to [\mathcal{P}']$, we define the extended mapping $\bar{\phi} : [\mathcal{P}] \to [\mathcal{P}']$ by $\bar{\phi}(A) := \bigcup\{\phi(C) | C \in \mathcal{P}, C \subseteq A\}$ for all $A \in [\mathcal{P}]$. We say that $\mathbf{p}$ is a *belief type* if and only if (a) for every $p \in \mathbf{p}$ there is some Γ–partition with $p : [\mathcal{P}] \to \mathbb{R}$,[2] (b) for every Γ–partition there is a $p \in \mathbf{p}$ with domain $[\mathcal{P}]$, and (c) for all

[2] Again, the use of the real numbers as a scale is not essential.

Γ–partitions $\mathcal{P}$, $\mathcal{P}'$, every 1–1 mapping $\phi : \mathcal{P}' \to \mathcal{P}$ and every p with domain $[\mathcal{P}]$: if $p \in \mathbf{p}$, then $p \circ \bar{\phi} \in \mathbf{p}$. We refer to property (c) by saying that $\mathbf{p}$ is *closed under relabelling*. For any Γ–partition $\mathcal{P}$, we define $\mathbf{p}(\mathcal{P})$ as the set of all $p \in \mathbf{p}$ with $p : [\mathcal{P}] \to \mathbb{R}$. We call any $p \in \mathbf{p}(\mathcal{P})$ a *($\mathbf{p}$–)belief measure on* $\mathcal{P}$. Suppose we have Γ–partitions $\mathcal{P}$, $\mathcal{P}'$ with $\mathcal{P} \subseteq [\mathcal{P}']$ and $p \in \mathbf{p}(\mathcal{P})$, $p' \in \mathbf{p}(\mathcal{P}')$. We then say that p' *refines* p if and only if $p(A) = p'(A)$ for all $A \in [\mathcal{P}]$.[3]

We embody into our the concept of a decision rule a non–triviality condition and the requirement to yield identical results if consequences are refined in a way that does not affect any utilities. For any belief type $\mathbf{p}$, G is a *decision rule for* $\mathbf{p}$ if and only if (a) G is a function such that, for any Γ–partition $\mathcal{P}$, any $p \in \mathbf{p}(\mathcal{P})$ and any $u \in \mathbf{u}(\mathcal{P})$, G maps $\langle p, u \rangle$ into $\mathbb{R}$, (b) for every Γ–partition $\mathcal{P}$ and every $p \in \mathbf{p}(\mathcal{P})$ there are $u, u' \in \mathbf{u}(\mathcal{P})$ such that $G(p, u) > G(p, u')$ and (c) for any Γ–partitions $\mathcal{P}$ and $\mathcal{P}'$ with $\mathcal{P} \subseteq [\mathcal{P}']$, any $p \in \mathbf{p}(\mathcal{P})$, $p' \in \mathbf{p}(\mathcal{P}')$, $u \in \mathbf{u}(\mathcal{P})$ and $u' \in \mathbf{u}(\mathcal{P}')$, if p' refines p and $u(C) = u'(X)$ for all $C \in \mathcal{P}$ and all $X \in \mathcal{P}'$ with $X \subseteq C$, then $G(p, u) = G(p', u')$. We refer to property (b) by saying that G is *non–trivial* and to property (c) by saying that G is *invariant under empty refinements*. We define a set of consequences to be null relative to a belief measure p and a decision rule G exactly when the utility assignments to the consequences in the set make no difference for the value of $G(p, u)$. For any Γ–partition $\mathcal{P}$, any belief type $\mathbf{p}$, any decision rule G for $\mathbf{p}$, any $p \in \mathbf{p}(\mathcal{P})$ and any $A \in [\mathcal{P}]$, we say that A is *p, G–null* if and only if, for all $u, u' \in \mathbf{u}(\mathcal{P})$: If $u(C) = u'(C)$ for all $C \in \mathcal{P}$ with $C \subseteq -A$, then $G(p, u) = G(p, u')$. Any $A \in [\mathcal{P}]$ is *p, G–one* if and only if $-A$ is p, G–null (cf. Observation A.1).

$M = \langle \mathcal{P}, u, \mathcal{F}, \mathbf{p}, P, G \rangle$ is a (generalized) *decision–theoretic model* if and only if $\mathcal{P}$ is a Γ–partition, u is a utility on $\mathcal{P}$, $\mathcal{F} \subseteq \Phi$, $\mathbf{p}$ is a belief type, P is a function $P : \mathcal{F} \to \mathbf{p}(\mathcal{P})$, and G is a decision rule for $\mathbf{p}$. We define $\mathcal{P}_M$, u_M, $\mathcal{F}_M$, $\mathbf{p}_M$, P_M and G_M to be the entities such that $M = \langle \mathcal{P}_M, u_M, \mathcal{F}_M, \mathbf{p}_M, P_M, G_M \rangle$. We write $P_{M,f} := P_M(f)$ (for all $f \in \mathcal{F}_M$). The function $V_M : \mathcal{F}_M \to \mathbb{R}$ defined by $V_M(f) := G_M(u_M, P_{M,f})$ (for all $f \in \mathcal{F}_M$) is the *evaluation function associated with M*.

Let $I \geq 2$ ($I \in \mathbb{N}$) be the fixed number of individuals. Let $\mathbf{G}(I)$ be the set of all vectors $\langle M_i \rangle$ of decision–theoretic models such that $\mathcal{P}_{M_i} = \mathcal{P}_{M_j}$, $\mathcal{F}_{M_i} = \mathcal{F}_{M_j}$ (for all $1 \leq i, j \leq I$). For any $\langle M_i \rangle \in \mathbf{G}(I)$, let $\mathcal{P}_{\langle M_i \rangle} := \mathcal{P}_{M_1}$ and $\mathcal{F}_{\langle M_i \rangle} := \mathcal{F}_{M_1}$. This definition allows the models in a vector $\langle M_i \rangle \in \mathbf{G}(I)$ to have different belief types and decision rules.

Generalized ex post social choice rules

A social choice rule yields a group choice function based on the decision–theoretic models that describe the individuals. For any $\mathcal{F} \subseteq \Phi$, C is a *choice function for* $\mathcal{F}$ if and only if $C : (2^{\mathcal{F}} - \{\varnothing\}) \to (2^{\mathcal{F}} - \{\varnothing\})$ and $C(X) \subseteq X$ for any $X \subseteq \mathcal{F}$. $M = \langle \mathcal{F}, C \rangle$

[3] Note that this notion of belief refinement is transitive: If p' refines p and p'' refines p', then p'' refines p. Our proofs work for any transitive notion of belief refinements.

is a *choice model* if and only if $\mathcal{F} \subseteq \Phi$ and C is a choice function on $\mathcal{F}$. We define $\mathcal{F}_M$ and C_M as the entities such that $M = \langle \mathcal{F}_M, C_M \rangle$. For arbitrary choice models $M = \langle \mathcal{F}, C \rangle$ and $M' = \langle \mathcal{F}', C' \rangle$, we say that M' *refines* M if and only if (i) $\mathcal{F} \subseteq \mathcal{F}'$ and (ii) $C(X) = C'(X)$ for all $X \subseteq \mathcal{F}$. We call s a *utility aggregation rule* if and only if, for any Γ–partition $\mathcal{P}$, s maps any vector $\langle u_i \rangle \in \mathbf{u}(\mathcal{P})^I$ to a choice function c on $\mathcal{P}$ (i.e., a function $c : (2^{\mathcal{P}} - \{\varnothing\}) \to (2^{\mathcal{P}} - \{\varnothing\})$ with $c(X) \subseteq X$ for any $X \subseteq \mathcal{P}$). We emphasize the generality of this concept. It subsumes the situation of our numerical example where real–valued one–dimensional individual utilities were aggregated into a real–valued one–dimensional group utility. Trivially, any group utility and any acyclical group preference generates a group choice function over consequences. Our concept of a utility aggregation rule avoids any rationality assumptions like the weak axiom of revealed preference.

We need only very general assumptions about the manner in which individual beliefs enter an ex post social choice rule. Our general definition of an ex post social choice rule altogether avoids mentioning any belief aggregation rule and, instead, uses the following unanimity concept. For any $\langle M_i \rangle \in \mathbf{G}(I)$, any $A \in [\mathcal{P}_{\langle M_i \rangle}]$ and any $f \in \mathcal{F}_{\langle M_i \rangle}$, we say that A is *unanimously f–one in* $\langle M_i \rangle$ if and only if, for all $1 \leq i \leq I$, A is $P_{M_i,f}, G_{M_i}$–one. For $f, g \in \Phi$, we say that f *absolutely dominates* g in $\langle M_i \rangle$ w.r.t. s if and only if there are $A, B \in [\mathcal{P}_{\langle M_i \rangle}]$ such that A is unanimously f–one in $\langle M_i \rangle$, B is unanimously g–one in $\langle M_i \rangle$ and $[s(\langle u_{M_i} \rangle)](\{D, C\}) = \{C\}$ for all $C \subseteq A$ and all $D \subseteq B$ ($C, D \in \mathcal{P}_{\langle M_i \rangle}$). The following definition merely requires ex post social choice rules to yield choice functions over the same set of acts that is evaluated by the individuals (clause 2), to evaluate consequences by aggregating individual utilities in a way that is not contaminated by beliefs or evaluations of acts and, in binary choices, to choose absolutely dominant acts in a sense that is much weaker than the sure–thing principle or related dominance conditions (clause 3).

Definition 2.1 *S is an* ex post *social choice rule if and only if (1) there is some non–empty set $\mathcal{G} \subseteq \mathbf{G}(I)$ such that $S : \mathcal{G} \to \mathbf{C}$, (2) for all $\langle M_i \rangle \in \mathcal{G}$, we have $\mathcal{F}_{S(\langle M_i \rangle)} = \mathcal{F}_{\langle M_i \rangle}$ and (3) there is some cardinal–ordinal utility aggregation rule s such that for all $\langle M_i \rangle \in \mathcal{G}$ and all $f, g \in \mathcal{F}_{\langle M_i \rangle}$: If f absolutely dominates g in $\langle M_i \rangle$ w.r.t. s, then $C_{S(\langle M_i \rangle)}(\{f, g\}) = \{f\}$.*

It is easy to verify the consistency of this definition. Let $\mathcal{G}_S$ be the set such that $S : \mathcal{G}_S \to \mathbf{R}$. We call any s that satisfies condition (3) *a utility aggregation rule associated with S*. An ex post social choice rule S *has a wide domain* if and only if, for any $\langle M_i^* \rangle \in \mathcal{G}_S$, any Γ–partition $\mathcal{P}$, any $\mathcal{F} \subseteq \Phi$, any $\langle P_i \rangle$ with $P_i : \mathcal{F} \to \mathbf{p}_{\langle M_i^* \rangle}(\mathcal{P})$ $(1 \leq i \leq I)$ and any $\langle u_i \rangle \in \mathbf{u}(\mathcal{P})^I$, there is some $\langle M_i \rangle \in \mathcal{G}_S$ such that, for all $1 \leq i \leq I$, $\mathcal{P}_{\langle M_i \rangle} = \mathcal{P}$, $u_{M_i} = u_i$, $\mathcal{F} \subseteq \mathcal{F}_{\langle M_i \rangle}$ and $P_{M_i,f} = P_{i,f}$ for all $f \in \mathcal{F}$. The clause $\mathcal{F} \subseteq \mathcal{F}_{\langle M_i \rangle}$ allows the underlying structure of acts to force the inclusion of certain acts into the models $\langle M_i \rangle$. In Savage's framework, for example, we have to consider as acts all functions from worlds to consequences and are not free to choose some smaller set $\mathcal{F}$.

Instability theorem

We now formulate the remaining key condition of 'one–refinability' which will supply us with a structure that is rich enough for our purpose. A belief type is one–refinable relative to a decision rule just in case we can elect any distinction introduced by a fine–grained model to be considered one. If the fine–grained model splits each consequence $C \in \mathcal{P}$ of the coarse model into two consequences C_1, C_2, then we can find a belief measure relative to which the set $\{C_1 | C \in \mathcal{P}\}$ is considered one. In order to state our requirement formally, we need two auxiliary definitions that concern a function ϕ that associates coarse–grained with fine–grained consequences. The first definition ensures that the function ϕ is compatible with the way in which $\mathcal{P}'$ refines $\mathcal{P}$. We say that $\mathcal{P}'$ *ϕ–refines* $\mathcal{P}$ if and only if $\mathcal{P}$ and $\mathcal{P}'$ are Γ–partitions with $\mathcal{P} \subseteq [\mathcal{P}']$ and ϕ is such that $\phi : \mathcal{P} \rightarrow \mathcal{P}'$ and $\phi(C) \subseteq C$ for all $C \in \mathcal{P}$. Our second auxiliary definition states that the set of fine–grained consequences selected by ϕ are treated as p', G–one. Suppose that G is a decision rule for $\mathbf{p}$ and $p, p' \in \mathbf{p}$. We then say that $\langle \mathcal{P}', p' \rangle$ *one$_\phi$–refines* $\langle \mathcal{P}, p \rangle$ w.r.t. G if and only if $\mathcal{P}'$ ϕ–refines $\mathcal{P}$, $p \in \mathbf{p}(\mathcal{P})$, $p' \in \mathbf{p}(\mathcal{P}')$, p' refines p, and $\bigcup\{\phi(C) | C \in \mathcal{P}\}$ is p', G–one. Finally, we can state the definition we had in mind. A belief type $\mathbf{p}$ is *one–refinable* w.r.t. G if and only if, for all $\mathcal{P}$, $\mathcal{P}'$ and ϕ such that $\mathcal{P}'$ ϕ–refines $\mathcal{P}$ and for all $p \in \mathbf{p}(\mathcal{P})$, there exists some $p' \in \mathbf{p}(\mathcal{P}')$ such that $\langle \mathcal{P}', p' \rangle$ one$_\phi$–refines $\langle \mathcal{P}, p \rangle$ w.r.t. G. We note that probabilities and capacities are clearly one–refinable w.r.t. maximizing (Choquet–)expected utility. We say that an ex post social choice rule S is *for one–refinable models* if and only if, for every $\langle M_i \rangle \in \mathcal{G}_S$, $\mathbf{p}_{M_i}$ is one–refinable w.r.t. G_{M_i} $(1 \leq i \leq I)$.

After formulating several properties of utility aggregation rules, we will state our instability theorem. A Γ–partition is *refinable* if and only if every $C \in \mathcal{P}$ has at least two elements. Suppose s is a cardinal–ordinal utility aggregation rule. s is *non–exceptional* if and only if there exists some refinable Γ–partition $\mathcal{P}$, (possibly identical) consequences $C_1, \ldots, C_I, D_1, \ldots, D_I \in \mathcal{P}$ and $\langle u_i \rangle, \langle u'_i \rangle \in \mathbf{u}(\mathcal{P})^I$ such that for any $1 \leq i, j, k \leq I$) $u_i(C_i) = u'_i(C_i)$ and $u_i(D_i) = u'_i(D_i)$, but $[s(\langle u_i \rangle)](\{C_j, D_k\}) = \{C_j\}$ and $[s(\langle u'_i \rangle)](\{C_j, D_k\}) = \{D_k\}$. For concreteness, we mention already the special case in which s' aggregates the individuals' real–valued one–dimensional utilities on $\mathcal{P}$ into a real–valued one–dimensional group utility on $\mathcal{P}$ and where s is the utility aggregation rule generated by s' (i.e., s yields the choice function generated by the group utility aggregated by s'). If s' is a utilitarian rule with weights λ_i, then s is non–exceptional if and only if the weights of at least two individuals are non–zero. By Observation A.5 in the appendix, a utility aggregation rule must be non–exceptional if it is Pareto optimal (cf. below). s is *independent of irrelevant alternatives (IIA)* if and only if, if $u_i(C) = u'_i(C)$ for all $1 \leq i \leq I$ and all $C \in X$, then $[s(\langle u_i \rangle)](X) = [s(\langle u'_i \rangle)](X)$ (for any Γ–partitions $\mathcal{P}$ and $\mathcal{P}'$, any $X \subseteq \mathcal{P} \cap \mathcal{P}'$ and any $\langle u_i \rangle \in \mathbf{u}(\mathcal{P})^I$ and $\langle u'_i \rangle \in \mathbf{u}(\mathcal{P}')^I$). Note that IIA forces the aggregation of utilities to be independent of the fine–graining of the consequence partition. When, as is usual in the literature, the consequence partition is held fixed, this aspect of IIA cannot be expressed. Note, moreover, that IIA for utility aggregation rules (which deliver a group choice function over consequences) is an extremely weak and ubiquitous condition since we do not presuppose the existence of a group preference or an axiom of revealed preference (cf. the applications in the following section and Plott, 1976). s is (ex post) *Pareto optimal* if and only if $[s(\langle u_i \rangle)](\{C, D\}) = \{C\}$ when $u_i(C) \geq u_i(D)$ for all $1 \leq i \leq I$ but $u_j(C) > u_j(D)$ for some $1 \leq j \leq I$ (for any Γ–partition $\mathcal{P}$, any $C, D \in \mathcal{P}$ and any $\langle u_i \rangle \in \mathbf{u}(\mathcal{P})^I$).

In a refinement, we now require not only preferences but also the evaluation functions and beliefs to remain unchanged. For any decision-theoretic models M and M' with $\mathbf{p}_M = \mathbf{p}_{M'}$ and $G_M = G_{M'}$, we say that M' *refines* M if and only if (1) $\mathcal{P}_M \subseteq [\mathcal{P}_{M'}]$, (2) $\mathcal{F}_M \subseteq \mathcal{F}_{M'}$ and, for all $f \in \mathcal{F}_M$, (3) $P_{M',f}$ refines $P_{M,f}$ and (4) $V_M(f) = V_{M'}(f)$. This definition is motivated by our interest in the instability theorem. In different contexts, we might wish to speak of an individual refinement already when individual preferences are left unchanged (replacing clause (4) of the definition of individual refinements by the condition that $V_{M'}(f) \geq V_{M'}(g)$ iff $V_{M'}(f) \geq V_{M'}(g)$ for all $f, g \in \mathcal{F}$). With our stronger definition, however, we will obtain a strong instability result. Our proofs will construct a violation of stability even for individual refinements in which evaluations remain unchanged. A fortiori, there then are violations of stability for individual refinements in the weaker sense.

Theorem 2.2 *Suppose S is an ex post social choice rule that has a wide domain and that is for one–refinable models with an associated utility aggregation rule that is (1) IIA and non–exceptional, or (2) Pareto optimal.*

Then there is an infinite sequence $\langle M_i^n \rangle_{n \in \mathbb{N}}$ of vectors of decision–theoretic models in $\mathcal{G}_S$ such that $\langle M_i^{n+1} \rangle$ refines $\langle M_i^n \rangle$ (for all $n \in \mathbb{N}$) and S leads to a sequence of group models $\langle S(\langle M_i^n \rangle) \rangle_{n \in \mathbb{N}}$ that oscillates between absolute dominance of f over g and absolute dominance of g over f (for some $f, g \in \mathcal{F}_{\langle M_i^0 \rangle}$). Hence, S is not stable under refinements.

The proof of this theorem uses the same technique as Hild (2001a, 2001b) and draw on extreme disagreements of the individuals' belief measures.

Applications

Hild (2001a) discusses utility aggregation rules that satisfy the preconditions of our Instability Theorems. Individuals may be described by any of the decision theory listed in Section 1. Any of these models is one–refinable and a decision–theoretic model in the sense of our definition.

We comment briefly on a special class of ex post social choice rule that generates social preferences by some decision theory on the list of Section 1 using aggregated beliefs and aggregated utilities. Assume that individual beliefs are represented either by subjective probabilities or, more generally, by capacities.[4] Choose any decision–theoretic model

[4]We say that p is a *probability on* $\mathcal{P}$ if and only if $p : [\mathcal{P}] \to \mathbb{R}$, $p(\varnothing) = 0$, $p(\Gamma) = 1$ and $p(A \cup B) = p(A) + p(B)$ for all $A, B \in [\mathcal{P}]$ with $A \cap B = \varnothing$. p is a *capacity on* $\mathcal{P}$ if and only if $p : [\mathcal{P}] \to \mathbb{R}$, $p(\varnothing) = 0$, $p(\Gamma) = 1$ and p is monotonic w.r.t. set inclusion. $C(p, u) := \int_0^\infty p(u \geq x)\, dx + \int_{-\infty}^0 [p(u \geq x) - 1]\, dx$ is the Choquet–expectation of u w.r.t. p.

on our list as a decision theory for the group. When this theory requires aggregated probabilities, let T aggregate capacities $\langle p_i \rangle$ on $\mathcal{P}$ into a probability on $\mathcal{P}$. When the group's decision theory requires only aggregated capacities, let T aggregate capacities $\langle p_i \rangle$ on $\mathcal{P}$ into a capacity on $\mathcal{P}$.[5] Let s' cardinal–cardinal utility aggregation rule that aggregates utilities on $\mathcal{P}$ into a utility on $\mathcal{P}$. We say that T *satisfies certainty agreement* if and only if, for Γ–partition $\mathcal{P}$, any $A \in [\mathcal{P}]$ and any vector $\langle p_i \rangle$ of capacities on $\mathcal{P}$, if $p_i(B) = p_i(B \cap A)$ for all $B \in [\mathcal{P}]$ and all $1 \leq i \leq I$, then $T(\langle p_i \rangle)(B) = T(\langle p_i \rangle)(B \cap A)$ for all $B \in [\mathcal{P}]$. In case p_i is a probability, we have $p_i(B) = p_i(B \cap A)$ for all $B \in [\mathcal{P}]$ exactly when $p_i(A) = 1$. Now assume that the group applies its decision theory to the aggregated utility and the aggregated belief measure (assuming that T yields probabilities when required by the decision theory). The social choice rule thus defined satisfies Definition 2.1 (cf. Observation A.6).[6] The conclusions of the ex post instability theorems then apply if s' has the properties specified in the theorems.

3 Robust Aggregation

We finally turn to Levi's (1990) robust mode of aggregation. In terms of instabilities, the situation for the robust mode is even worse than for the ex post mode: Any non–trivial robust aggregation is unstable. The idea behind robust aggregation is to consider not only the individuals' actual preferences but also what we can call their 'empathetic' preferences. Empathetic preferences are the preferences that real individuals would have if they were to keep their own utilities but adopted another individual's probabilities. We now assume that all $\langle M_i \rangle \in \mathbf{G}(I)$ have the same belief type and decision rule ($\mathbf{p}_{M_i} = \mathbf{p}_{M_j}$ and $G_{M_i} = G_{M_j}$ for $1 \leq i, j \leq I$). For $1 \leq k, l \leq I$ and $\langle M_i \rangle \in \mathbf{G}(I)$ we define M_{kl} as the evaluation model such that $\mathcal{F}_{M_{kl}} = \mathcal{F}_{\langle M_i \rangle}$ and $V_{M_{kl}}(f) = G_{\langle M_i \rangle}(P_{M_k,f}, u_{M_l})$ for all $f \in \mathcal{F}_{\langle M_i \rangle}$. Note that M_i denotes a decision–theoretic model while M_{ii} denotes an evaluation model. An evaluation model M_{kl} contains the evaluation function of a hypothetical individual with individual k's beliefs and l's utility. There are I^2 such hypothetical individuals. For any $\langle M_i \rangle \in \mathbf{G}(I)$, we write $f \succeq_{M_{ij}} g$ if and only if $V_{M_{ij}}(f, g) \succeq V_{M_{ij}}(g, f)$ (for all $1 \leq i, j \leq I$ and $f, g \in \mathcal{F}_{\langle M_i \rangle}$).

Let $\mathbf{V}(I)$ be the set of all vectors $\langle \mathcal{F}_i, V_i \rangle$ of evaluation models with $\mathcal{F}_i = \mathcal{F}_j \subseteq \Phi$ and $V_i : \mathcal{F}_i \to \mathbb{R}$ ($1 \leq i, j \leq I$). For $\langle M_i \rangle \in \mathcal{V}(I)$, let $\mathcal{F}_{\langle M_i \rangle} := \mathcal{F}_{M_1}$. S' is an *ex ante social choice rule* if and only if there is some non–empty set $\mathcal{V} \subseteq \mathbf{V}(I)$ such that $S' : \mathcal{V} \to \mathbf{C}$ and $\mathcal{F}_{S(\langle M_i \rangle)} = \mathcal{F}_{\langle M_i \rangle}$ for all $\langle M_i \rangle \in \mathcal{V}$. S is a *robust social choice rule* if and only if there is some non–empty set $\mathcal{G} \subseteq \mathbf{G}(I)$ with $S : \mathcal{G} \to \mathbf{C}$ and some ex ante social choice rule $S' : \mathbf{V}(I^2) \to \mathbf{C}$ such that $S(\langle M_i \rangle) = S'(\langle M_{kl} \rangle)$ for all $\langle M_i \rangle \in \mathcal{G}$. Any S' with this

[5] Clearly, we can apply T even if individual beliefs are represented by probabilities, since probabilities are also capacities.

[6] In the case of regret theory, the group's modification function must be regular. In the case of expected utility with threshold $\alpha_0 \in \mathbb{R}$ and an aggregated group utility u_0, we define the group's preference r over $\mathcal{P}$ such that $C \succ_r D$ iff $u_0(C) - u_0(D) > \alpha_0$ (for any $C, D \in \mathcal{P}$). Then clauses 3a and 3b of Definition 2.1 are satisfied.

property is *an ex ante rule associated with* S.[7] If S is a robust social choice rule, let $\mathcal{G}_S$ be the set such that $S : \mathcal{G}_S \to \mathbf{R}$.

We now formulate several additional properties of a robust social choice rule S. S is *non–trivial* if and only if there are $\langle M_i\rangle, \langle M_i^*\rangle \in \mathcal{G}_S$ such that $\mathcal{F}_{\langle M_i\rangle} = \mathcal{F}_{\langle M_i^*\rangle}$, $\langle V_{M_i}\rangle = \langle V_{M_i^*}\rangle$ but $S(\langle M_i\rangle) \neq S(\langle M_i^*\rangle)$. S is *independent of irrelevant alternatives (IIA)* if and only if S has some associated ex ante rule $S' : \mathcal{V} \to \mathbf{C}$ such that, for all $\langle \mathcal{F}, V_i\rangle, \langle \mathcal{F}', V_i'\rangle \in \mathcal{V}$ and all $X \subseteq \mathcal{F} \cap \mathcal{F}'$, if $V_i|X = V_i'|X$ for all $1 \leq i \leq I^2$, then $C_{S'(\langle M_i\rangle)}(X) = C_{S'(\langle M_i'\rangle)}(X)$. S is *Pareto optimal* if and only if, for all $\langle M_i\rangle$ in the domain of S and all $f, g \in \mathcal{F}_{\langle M_i\rangle}$, if $f \succeq_{M_{ij}} g$ for all $1 \leq i, j \leq I$ but $f \succ_{M_{kl}} g$ for some $1 \leq k, l \leq I$, then $C_{S(\langle M_i\rangle)}(\{f, g\}) = \{f\}$. S is *has a wide domain* if and only if for any $\langle M_i^*\rangle \in \mathcal{G}_S$, any Γ–partition $\mathcal{P}$, any $\mathcal{F} \subseteq \Phi$, any $\langle P_i\rangle$ with $P_i : \mathcal{F} \to \mathbf{p}_{\langle M_i^*\rangle}(\mathcal{P})$ (for all $1 \leq i \leq I$) and any $\langle u_i\rangle \in \mathbf{u}(\mathcal{P})^I$, there is some $\langle M_i\rangle \in \mathcal{G}_S$ such that, for all $1 \leq i \leq I$, $\mathcal{P}_{\langle M_i\rangle} = \mathcal{P}$, $u_{M_i} = u_i$, $\mathcal{F} \subseteq \mathcal{F}_{\langle M_i\rangle}$, and $P_{M_i,f} = P_{i,f}$ for all $f \in \mathcal{F}$.

Finally, we formulate two additional properties of belief types and decision rules. We will require that, for suitable two–dimensional partitions, we can find a belief measure that delivers any arbitrary marginal belief measures. We will require decision rules to depend only on the values of belief measures and utilities and on no other features of consequences. Formally, a belief type $\mathbf{p}$ is *cross–product refinable* if and only if for all Γ–partitions $\mathcal{P}$, $\mathcal{P}^*$ and $\mathcal{Q}$ and for all $p \in \mathbf{p}(\mathcal{P})$, $p^* \in \mathbf{p}(\mathcal{P}^*)$, if $X \cap X^* \in \mathcal{Q}$ for all $X \in \mathcal{P}$, $X^* \in \mathcal{P}^*$,[8] then there exists a $q \in \mathbf{p}(\mathcal{Q})$ that refines p and refines p^*. Both probabilities and capacities are cross–product refinable (cf. Observation A.6). A decision rule G for $\mathbf{p}$ is *invariant under relabelling* if and only if for all Γ–partitions $\mathcal{P}$ and $\mathcal{P}'$, any 1–1 mapping $\phi : \mathcal{P}' \to \mathcal{P}$ and any $p \in \mathbf{p}(\mathcal{P})$ and $u \in \mathbf{u}(\mathcal{P})$, we have $G(p, u) = G(p \circ \bar{\phi}, u \circ \phi)$. All decision theories listed in the introduction have this property. Finally, a robust social choice rule S is *for models that are one–refinable, cross–product refinable and invariant under relabelling* if and only if, for every $\langle M_i\rangle \in \mathcal{G}_S$, $\mathbf{p}_{\langle M_i\rangle}$ cross–product refinable and one–refinable w.r.t. $G_{\langle M_i\rangle}$ and $G_{\langle M_i\rangle}$ is invariant under relabelling.

Theorem 3.1 *Suppose that S is a robust social choice rule that has a wide domain is for models that are one–refinable, cross–product refinable and invariant under relabelling.*

If S is non–trivial, then it cannot be stable under refinements.

Theorem 3.2 *Suppose that S is a robust social choice rule that has a wide domain and is for one–refinable. Suppose S is (1) IIA, non–trivial and for models that are cross–product refinable and invariant under relabelling, or (2) Pareto optimal.*

Then there is an infinite sequence $\langle M_i^n\rangle_{n\in\mathbb{N}}$ of vectors of decision–theoretic models in $\mathcal{G}_S$ such that $\langle M_i^{n+1}\rangle$ refines $\langle M_i^n\rangle$ (for all $n \in \mathbb{N}$) and S leads to a sequence of group models $\langle S(\langle M_i^n\rangle)\rangle_{n\in\mathbb{N}}$ that, for some $f \in \mathcal{F}_{\langle M_i^0\rangle}$, oscillates between choosing and not

[7] There are generally several ex ante rule associated with S when S has a restricted domain.

[8] By the definition of a partition (footnote 2), this implies that $X \cap X^* \neq \varnothing$ for any $X \in \mathcal{P}$ and $X^* \in \mathcal{P}^*$.

choosing f from some choice set $X \subseteq \mathcal{F}_{\langle M_i^0 \rangle}$ (i.e., there is some $X \subseteq \mathcal{F}_{\langle M_i^0 \rangle}$ and $f \in X$ with $f \in C_{S(\langle M_i^{2n} \rangle)}(X)$ and $f \notin C_{S(\langle M_i^{2n+1} \rangle)}(X)$ for all $n \in \mathbb{N}$).

We already noted that probabilities and capacities are cross–product refinable. Hence, Theorem 3.1 and Theorem 3.2 apply to individuals who are described by any of the decision theories listed in Section 1.

Appendix A Proofs: Ex Post Aggregation

Lemmata and Observations

For any $u, u' \in \mathbf{u}(\mathcal{P})$ and any $A \in [\mathcal{P}]$, we write $u =_A u'$ if and only if $u(C) = u'(C)$ for all $C \in \mathcal{P}$ with $C \subseteq A$.

Observation A.1 *Suppose* $\mathbf{p}$ *is a belief type and* G *is a decision rule for* $\mathbf{p}$*. Suppose furthermore that* $\mathcal{P}$ *is a* Γ*–partition,* $p \in \mathbf{p}(\mathcal{P})$ *and* $A, B \in [\mathcal{P}]$*. Then the following holds:*
1.) Γ *is not* p, G*–null.*
2.) If A *and* B *are* p, G*–null, then* $A \cup B$ *is* p, G*–null.*
3.) If A *is* p, G*–null, then* $-A$ *is not* p, G*–null.*
4.) If A *is* p, G*–null and* $B \subseteq A$*, then* B *is* p, G*–null.*

Proof: 1.) Because G is non–trivial. 2.) Suppose that A and B are p, G–null. Take any $u, u' \in \mathbf{u}(\mathcal{P})$ with $u =_{-(A\cup B)} u'$ and define $w :=_{-A} u$ and $w :=_A u'$. Hence, $w \in \mathbf{u}(\mathcal{P})$. Since $w =_{-B\cap A} u'$ and $w =_{-B\cap -A} u =_{-B\cap -A} u'$, we have $w =_{-B} u'$. Hence, $G(p, u) = G(p, w) = G(p, u')$ for any $u, u' \in \mathbf{u}(\mathcal{P})$ with $u =_{-(A\cup B)} u'$. 3.) Suppose that A and $-A$ are p, G–null. By (2), Γ is p, G–null but, by (1) Γ is not p, G–null. Contradiction! 4.) Suppose that A is p, G–null and $B \subseteq A$. Then, for all $u, u' \in \mathbf{u}(\mathcal{P})$ with $u =_{-A} u'$, we have $G(p, u) = G(p, u')$. Since $-A \subseteq -B$, we obtain $G(p, u) = G(p, u')$ for all $u, u' \in \mathbf{u}(\mathcal{P})$ with $u =_{-B} u'$. $\square$

Lemma A.2 *Suppose that* $\mathbf{p}$ *is a belief type with* $p, p' \in \mathbf{p}$ *and* G *is a decision rule for* $\mathbf{p}$*. Suppose furthermore that* $\langle \mathcal{P}', p' \rangle$ *one*$_\phi$*–refines* $\langle \mathcal{P}, p \rangle$ *w.r.t.* G*. Then, for any* $A \in [\mathcal{P}]$*,* $\bar{\phi}(A)$ *is* p', G*–one whenever* A *is* p, G*–one.*

Proof: Suppose that $A \in [\mathcal{P}]$ is p, G–one and $u, u' \in \mathbf{u}(\mathcal{P}')$ with $u =_{\bar{\phi}(A)} u'$. Define $u^*, u^{**} \in \mathbf{u}(\mathcal{P})$ by $u^* := u \circ \phi$ and $u^{**} := u' \circ \phi$. Since $u^* =_A u^{**}$ and A is p, G–one, we have $G(p, u^*) = G(p, u^{**})$. Next, define $u^\dagger, u^{\dagger\dagger} \in \mathbf{u}(\mathcal{P}')$ by $u^\dagger(X) := u^*(C)$ and $u^{\dagger\dagger}(X) := u^{**}(C)$ for all $C \in \mathcal{P}$, and $X \in \mathcal{P}'$ with $X \subseteq C$. Since G is invariant under empty refinements, $G(p, u^*) = G(p', u^\dagger)$ and $G(p, u^{**}) = G(p', u^{\dagger\dagger})$. Since $u^\dagger =_{\bar{\phi}(\Gamma)} u$, $u^{\dagger\dagger} =_{\bar{\phi}(\Gamma)} u'$ and $\bar{\phi}(\Gamma)$ is p', G–one (because $\langle \mathcal{P}', p' \rangle$ one$_\phi$–refines $\langle \mathcal{P}, p \rangle$ w.r.t. G), we have $G(p', u^\dagger) = G(p', u)$ and $G(p', u^{\dagger\dagger}) = G(p', u')$. Hence, $G(p', u) = G(p', u')$ whence $\bar{\phi}(A)$ is p', G–one. $\square$

We say that a decision rule G for $\mathbf{p}$ is *invariant under one–refinements* if and only if, for all $\mathcal{P}$, $\mathcal{P}'$, $p, p' \in \mathbf{p}$ and ϕ such that $\langle \mathcal{P}', p' \rangle$ one$_\phi$–refines $\langle \mathcal{P}, p \rangle$ w.r.t. G and for all $u \in \mathbf{u}(\mathcal{P})$, $u' \in \mathbf{u}(\mathcal{P}')$ and all $A \in [\mathcal{P}]$, if A is p, G–one and $u =_A u' \circ \phi$, then $G(p, u) = G(p', u')$.

Lemma A.3 *Any decision rule is invariant under one–refinements.*

Proof: Let G be a decision rule for $\mathbf{p}$ and let $p, p' \in \mathbf{p}$. Suppose that $\langle \mathcal{P}', p' \rangle$ one$_\phi$–refines $\langle \mathcal{P}, p \rangle$ w.r.t. G and that $u \in \mathbf{u}(\mathcal{P})$, $u' \in \mathbf{u}(\mathcal{P}')$. Suppose furthermore that $A \in [\mathcal{P}]$ is p, G–one and $u =_A u' \circ \phi$. Let $u^* \in \mathbf{u}(\mathcal{P}')$ be the function such that $u^*(X) := u(C)$ for all $X \in \mathcal{P}'$, $C \in \mathcal{P}$ and $X \subseteq C$. Because G is invariant under empty refinements, $G(p, u) = G(p', u^*)$. On the other hand, $G(p', u^*) = G(p', u')$ because $u^* =_{\bar{\phi}(A)} u'$ and $\bar{\phi}(A)$ is p', G–one (by Lemma A.2). Hence, $G(p, u) = G(p', u')$. $\square$

Lemma A.4 *Suppose* G *is a decision rule for* $\mathbf{p}$ *and* $\mathbf{p}$ *is one–refinable. Then for every* Γ*–partition* $\mathcal{P}$ *and every* $C \in \mathcal{P}$*, there exists some* $p \in \mathbf{p}(\mathcal{P})$ *such that* C *is* p, G*–one.*

Proof: Let $\mathcal{P}$ an arbitrary Γ–partition and $C \in \mathcal{P}$. Let $\phi : \{\Gamma\} \to \mathcal{P}$ be defined by $\phi(\Gamma) := C$. Obviously, $\mathcal{P}$ ϕ–refines $\{\Gamma\}$. Let $p^* \in \mathbf{p}(\Gamma)$. Since $\mathbf{p}$ is one–refinable, there is a $p \in \mathbf{p}(\mathcal{P})$ such that $\phi(\Gamma) = C$ is p, G–one. $\square$

Lemma A.5 *Suppose s is a cardinal–ordinal utility aggregation rule that is (1) IIA and non–exceptional, or (2) Pareto optimal. Then there exist* $v^i_j, w^i_j \in Z_i$ *with* $v^i_i = w^i_i$ *and* $v^i_{i+I} = w^i_{i+I}$ $(1 \leq i \leq I,$ $1 \leq j \leq 2I)$ *such that for all* Γ*–partitions* $\mathcal{P}$ *with* $2 \cdot I$ *different consequences* $Y_1, \ldots, Y_{2I} \in \mathcal{P}$ *and for all* $\langle u_i \rangle, \langle u'_i \rangle \in \mathbf{u}(\mathcal{P})^I$, $u_i(Y_j) = v^i_j$ *and* $u'_i(Y_j) = w^i_j$ *(for all* $1 \leq i \leq I$, $1 \leq j \leq 2I$*), we have* $[s(\langle u_i \rangle)](\{Y_l, Y_m\}) = \{Y_l\}$ *but* $[s(\langle u'_i \rangle)](\{Y_l, Y_m\}) = \{Y_m\}$ *(for all* $1 \leq l \leq I$ *and* $I+1 \leq m \leq 2I$*).*

Proof: Cf. Hild (2001a)

Observation A.6 *Let* $\mathbf{p}$ *be the set of all capacity on all* Γ*–partition. For any (finite)* Γ*–partition* $\mathcal{P}$*, let* $\mathbf{p}(\mathcal{P})$ *be the set of all capacities* $\mathcal{P}$*. Let* $\mathbf{u}$ *be the set of all one–dimensional utilities. Let* $C(.,.)$ *be Choquet–expectation (cf. footnote* **4***). Then the following holds:*
1.) For any $p \in \mathbf{p}(\mathcal{P})$ *and* $A \in [\mathcal{P}]$*,* A *is* p, C*–one iff* $p(B) = p(B \cap A)$ *for all* $B \in [\mathcal{P}]$*.*
2.) $\mathbf{p}$ *is a) one–refinable and b) cross–product refinable.*
3.) C *is a decision rule (i.e., non–trivial and invariant under empty refinements).*
4.) For all $p \in \mathbf{p}(\mathcal{P})$ *and all* $A \in [\mathcal{P}]$ *such that* A *is* p, C*–one, we have* $\min\{u(C)|C \in \mathcal{P}, C \subseteq A\} \leq C(p,u) \leq \max\{u(C)|C \in \mathcal{P}, C \subseteq A\}$*, for any* $u \in \mathbf{u}(\mathcal{P})$*.*

Proof: For a fixed Γ–partition $\mathcal{P}$ and $u \in \mathbf{u}(\mathcal{P})$, we write '$\{u \geq x\}$' instead of '$\bigcup\{C \in \mathcal{P}|u(C) \geq x\}$' and we omit curly brackets where this is unambiguous. 1.) We first show 'if'. Suppose that $p(B) = p(B \cap A)$ for all $B \in [\mathcal{P}]$ and let $u, u' \in \mathbf{u}(\mathcal{P})$ with $u =_A u'$. We note that $\{u \geq x\}, \{u' \geq x\} \in [\mathcal{P}]$ and, hence, $p(u \geq x) = p(\{u \geq x\} \cap A)$ and $p(u' \geq x) = p(\{u' \geq x\} \cap A)$. Since $\{u \geq x\} \cap A = \{u' \geq x\} \cap A$, we obtain $p(u \geq x) = p(u' \geq x)$ and, thus, $C(p,u) = C(p,u')$. Therefore, A is p, C–one. We now show 'only if'. For any $X \in [\mathcal{P}]$ and define $\mathbf{1}_X \in \mathbf{u}(\mathcal{P})$ as follows: $\mathbf{1}_X(Y) = 1$ if $Y \subseteq X$ and $\mathbf{1}_X(Y) = 0$ otherwise (for all $Y \in \mathcal{P}$). Assume that A is p, C–one and $B \in [\mathcal{P}]$. Clearly, $\mathbf{1}_B =_A \mathbf{1}_{B \cap A}$. Hence, $C(p, \mathbf{1}_B) = C(p, \mathbf{1}_{B \cap A})$. This implies $\int_0^\infty p(\mathbf{1}_B \geq x)\, dx = \int_0^\infty p(\{\mathbf{1}_{B \cap A} \geq x\}\, dx$ and $p(B) = p(B \cap A)$.
2.a) Suppose that p is a capacity on $\mathcal{P}$ and $\mathcal{P}'$ ϕ–refines $\mathcal{P}$. Recall that $\bar{\phi}$ is injective and has range $X := \bar{\phi}(\Gamma)$. We can, therefore, define $p'(B) := p(\bar{\phi}^{-1}(B \cap X))$ for all $B \in [\mathcal{P}']$. It follows that p' is a capacity on $\mathcal{P}'$, p' refines p (because $A = \bar{\phi}^{-1}(A \cap X)$ for all $A \in [\mathcal{P}]$), and X is p', C–one (because $p'(B \cap X) = p'(B)$ for all $B \in [\mathcal{P}']$). Thus, $\langle \mathcal{P}', p' \rangle$ one$_\phi$–refines $\langle \mathcal{P}, p \rangle$. 2.b) Suppose $\mathcal{P}$, $\mathcal{P}^*$ and $\mathcal{Q}$ are Γ–partitions such that $X \cap X^* \in \mathcal{Q}$ (for all $X \in \mathcal{P}$, $X^* \in \mathcal{P}^*$) and $p \in \mathbf{p}(\mathcal{P})$, $p^* \in \mathbf{p}(\mathcal{P}^*)$. By the definition of a partition (p. 2), we know that $X \cap X^* \neq \varnothing$ for any $X \in \mathcal{P}$ and $X^* \in \mathcal{P}^*$ and that for any $Y \in \mathcal{Q}$ there are $X \in \mathcal{P}$ and $X^* \in \mathcal{P}^*$ with $Y = X \cap X^*$ (since all sets of the form $X \cap X^*$ are mutually disjoint and their totality is jointly exhaustive of Γ). Similarly, (*) $A \cap A^* \in [\mathcal{Q}] - \{\varnothing\}$ for any $A \in [\mathcal{P}] - \{\varnothing\}$ and $A^* \in [\mathcal{P}^*] - \{\varnothing\}$. We say that A and A^* are *composers of* $C \in [\mathcal{Q}]$ if and only if $A \in [\mathcal{P}]$, $A^* \in [\mathcal{P}^*]$ and $C = A \cap A^*$. We say that $C \in [\mathcal{Q}]$ is *composable* if and only if there are composers of C; otherwise, we call C *non–composable*. Note that $\varnothing$ is non–composable. We show that (**) any composable $C \in [\mathcal{Q}]$ has unique composers. Suppose $A_1, A_2 \in [\mathcal{P}]$, $A^*_1, A^*_2 \in [\mathcal{P}^*]$ and $C = A_1 \cap A^*_1 = A_2 \cap A^*_2$. Since C is non–empty, A_1, A_2, A^*_1, A^*_2 also have to be non–empty. We can then express C, on the one hand, as the union of $(A_1 \cap -A_2) \cap A^*_1$ and $(A_1 \cap A_2) \cap A^*_1$ and, on the other hand, as the union of $(-A_1 \cap A_2) \cap A^*_2$ and $(A_1 \cap A_2) \cap A^*_2$. Since $(A_1 \cap -A_2)$ and $(-A_1 \cap A_2)$ are disjoint, we must have $(A_1 \cap -A_2) \cap A^*_1 = (-A_1 \cap A_2) \cap A^*_2 = \varnothing$. Since A^*_1, A^*_2 are non–empty, (*) implies that $(A_1 \cap -A_2) = (-A_1 \cap A_2) = \varnothing$ which is equivalent to $A_1 = A_2$. Analogously, we conclude that $A^*_1 = A^*_2$ and, thus, establish claim (**). We now define a capacity q on $\mathcal{Q}$ in three steps. Step 1: For any composable $C \in [\mathcal{Q}]$, let $q(C) := p(A) \cdot p^*(A^*)$ where $A \in [\mathcal{P}]$ and $A^* \in [\mathcal{P}^*]$ are the unique composers of C. Step 2: For any non–composable $B \in [\mathcal{Q}] - \{\varnothing\}$, let $q(B) := \max q(C)$ where the maximum is taken over all composable $C \in [\mathcal{Q}]$ with $C \subseteq B$. Step 3: $q(\varnothing) := 0$. For any composable $C_1, C_2 \in [\mathcal{Q}]$ with $C_1 \subseteq C_2$, we now have $q(C_1) \leq q(C_2)$ because the composers of C_1 must be subsets of the composers of C_2 and capacities are monotonic w.r.t. set inclusion. Hence, (***) for any composable $C \in [\mathcal{Q}]$, we have $q(C) = \max \sum q(C')$ where the maximum is taken over all composable $C' \in [\mathcal{Q}]$ with $C' \subseteq C$. We now show that q thus defined is indeed a capacity on $\mathcal{Q}$. By Step 3, $q(\varnothing) = 0$. By Step 1, $q(\Gamma) = p(\Gamma) \cdot p^*(\Gamma) = 1$. It remains to be shown that q is monotonic w.r.t. set inclusion. Suppose $B_1, B_2 \in [\mathcal{Q}]$ and $B_1 \subseteq B_2$. We only consider the case where $B_1, B_2 \neq \varnothing$ (all other cases are trivial). By (***), we know, on the one hand, that $q(B_1) = \max \sum q(C_1)$ where the maximum is taken over

all composable $C_1 \in [\mathcal{Q}]$ with $C_1 \subseteq B_1$ and, on the other hand, that $q(B_2) = \max \sum q(C_2)$ where the maximum is taken over all composable $C_2 \in [\mathcal{Q}]$ with $C_2 \subseteq B_2$. Since $B_1 \subseteq B_2$, every composable subset $C_1 \in [\mathcal{Q}]$ of B_1 is also a subset of B_2. Hence, $q(B_1) \leq q(B_2)$. Finally, we note that the composers of any $A \in [\mathcal{P}]$ are A and Γ while the composers of any $A^* \in [\mathcal{P}^*]$ are Γ and A^*. Hence, by Step 1, $q(A) = p(A) \cdot 1$ and $q(A^*) = 1 \cdot p^*(A^*)$. We thus establish that q refines both p and p^*. (We notice en passant that, by this construction, q is not a probability even if p and p^* are probabilities.) 3.) Trivial. 4.) Suppose $A \in [\mathcal{P}]$ is p, C–one, $u \in \mathbf{u}(\mathcal{P})$ and define $s := \max\{u(C) | C \in \mathcal{P}, C \subseteq A\}$. We only prove $C(p, u) \leq s$. The second half of the claim is analogous. We identify $s \in \mathbb{R}$ with the function $u' \in \mathbf{u}(\mathcal{P})$ that is constant $= s$. Define $v \in \mathbf{u}(\mathcal{P})$ by $v :=_A u$ and $v :=_{-A} s$. Since A is p, C–one, $C(p, u) = C(p, v)$. Since $v \leq s$, $C(p, v) \leq C(p, s)$ (by the monotonicity of $C(p, .)$. Hence, $C(p, u) \leq C(p, s) = s$. □

Theorems

For any $\langle M_i \rangle \in \mathbf{G}(I)$, any $A \in [\mathcal{P}_{\langle M_i \rangle}]$, any $f \in \mathcal{F}_{\langle M_i \rangle}$ and any $1 \leq i \leq I$, we say that A is *f–one in M_i* if and only if A is $P_{M_i,f}, G_{M_i}$–one. Note that in the following proofs there is an important notational difference between 'P_i' (a function P indexed by i) and 'P_{M_i}' (the function P occurring in M_i).

Theorem 2.2 *Proof:* Suppose S is an ex post social choice rule that has a wide domain and is for one–refinable models and s is a utility aggregation rule associated with S. Suppose furthermore that (1) s is IIA and non–exceptional, or (2) s is Pareto optimal. By assumption, Γ is a least countably infinite and, thus, we can partition it into $2 \cdot I$ different, at least countably infinite, sets $X_1, \ldots, X_{2I}$. For each $1 \leq k \leq 2I$, we can hence enumerate the elements of some countably infinite subset of X_k in a sequence $x_k^1, x_k^2, \ldots, x_k^n, \ldots$ (where $x_k^n \neq x_k^{n'}$ for any $n, n' \in \mathbb{N}^+$). We then define a sequence $\langle \mathcal{P}_k^n \rangle$ of partitions of X_k by $\mathcal{P}_k^n = \{\{x_k^1\}, \ldots, \{x_k^n\}, X_k - \{x_k^1, \ldots, x_k^n\}\}$ for all $n \in \mathbb{N}^+$. We write $X_k^0 := X_k$ and $X_k^n := X_k - \{x_k^1, \ldots, x_k^n\}$ for all $n \in \mathbb{N}^+$. Let $\mathcal{P}^0 := \{X_1, \ldots, X_{2I}\}$ and $\mathcal{P}^n := \mathcal{P}_1^n \cup \ldots \cup \mathcal{P}_{2I}^n$ for all $n \in \mathbb{N}^+$. Let $\phi^0 : \mathcal{P}^0 \to \mathcal{P}^1$ and $\psi^0 : \mathcal{P}^0 \to \mathcal{P}^1$ be defined by $\phi^0(X_k^0) := X_k^1$ and $\psi^0(X_k^0) := \{x_k^1\}$ (for all $1 \leq k \leq 2I$). For any $n \in \mathbb{N}^+$, let $\phi^n : \mathcal{P}^n \to \mathcal{P}^{n+1}$ and $\psi^n : \mathcal{P}^n \to \mathcal{P}^{n+1}$ be defined by $\phi^n(\{x_k^{n'}\}) := \{x_k^{n'}\}$ and $\psi^n(\{x_k^{n'}\}) := \{x_k^{n'}\}$ for all $n' \leq n$ ($n' \in \mathbb{N}^+$) and by $\phi^n(X_k^n) := X_k^{n+1}$ and $\psi^n(X_k^n) := \{x_k^{n+1}\}$ (for all $1 \leq k \leq 2I$). Trivially, $\mathcal{P}^{n+1}$ both ϕ^n–refines and ψ^n–refines $\mathcal{P}^n$ (for all $n \in \mathbb{N}$).

By Lemma A.5, there exists $\langle M_i^* \rangle \in \mathcal{G}_S$ and $v_j^i, w_j^i \in \mathbb{R}^L$ with $v_i^i = w_i^i$ and $v_{i+I}^i = w_{i+I}^i$ ($1 \leq i \leq I$, $1 \leq j \leq 2I$) such that for all Γ–partitions $\mathcal{P}$ with $2 \cdot I$ different consequences $Y_1, \ldots, Y_{2I} \in \mathcal{P}$ and for all $\langle u_i \rangle, \langle u_i' \rangle \in \mathbf{u}(\mathcal{P})^I$ with $u_i(Y_j) = v_j^i$, and $u_i'(Y_j) = w_j^i$ (for all $1 \leq i \leq I$, $1 \leq j \leq 2I$), we have $[s(\langle u_i \rangle)](\{Y_l, Y_m\}) = \{Y_l\}$ but $[s(\langle u_i' \rangle)](\{Y_l, Y_m\}) = \{Y_m\}$ (for all $1 \leq l \leq I$ and $I + 1 \leq m \leq 2I$). Let $\mathbf{p}_i := \mathbf{p}_{M_i^*}$, and $G_i := G_{M_i^*}$ (for all $1 \leq i \leq I$). We now recursively define a sequence $\langle u_1^n, \ldots, u_I^n \rangle_{n \in \mathbb{N}}$ of vectors of individual utility functions such that, for each $n \in \mathbb{N}$ and $1 \leq i \leq I$, $u_i^n \in \mathbf{u}(\mathcal{P}^n)$. For all even $n \in \mathbb{N}$, let $u_i^n(X_k^n) := v_k^i$ and, for all odd $n \in \mathbb{N}$, let $u_i^n(X_k^n) := w_k^i$ (for all $1 \leq i \leq I$ and all $1 \leq k \leq 2I$). Finally, for any $n, n' \in \mathbb{N}^+$ with $n' \leq n$, let $u_i^{n+1}(\{x_k^{n'}\}) := u_i^n(\{x_k^{n'}\})$ and, for any $n \in \mathbb{N}$, let $u_i^{n+1}(\{x_k^{n+1}\}) := u_i^n(X_k^n)$ (for all $1 \leq i \leq I$, and $1 \leq k \leq 2I$). It follows from Lemma A.5 that, for all $n \in \mathbb{N}$, (*) $[s(\langle u_i^{2n} \rangle)](\{X_l^{2n}, X_m^{2n}\}) = \{X_l^{2n}\}$ and $[s(\langle u_i^{2n+1} \rangle)](\{X_l^{2n+1}, X_m^{2n+1}\}) = \{X_m^{2n+1}\}$ (for all $1 \leq l \leq I$ and $I + 1 \leq m \leq 2I$). Moreover, (**) $u_i^n = u_i^{n+1} \circ \psi^n$ for any $n \in \mathbb{N}$ and $1 \leq i \leq I$. Next we note that, by assumption, there are $c, d \in \Phi$ with $c \neq d$. It can be shown by induction that there is a sequence $\langle M_1^n, \ldots, M_I^n \rangle_{n \in \mathbb{N}}$ of vectors of individual decision–theoretic models such that, for all $n \in \mathbb{N}$, $\langle M_1^n, \ldots, M_I^n \rangle \in \mathcal{G}_S$ and, for all $1 \leq i \leq I$, $\mathcal{P}_{\langle M_i^n \rangle} = \mathcal{P}^n$, $u_{M_i^n} = u_i^n$, $c, d \in \mathcal{F}_{\langle M_i^n \rangle} \subseteq \mathcal{F}_{\langle M_i^{n+1} \rangle}$, $\mathbf{p}_{M_i^n} = \mathbf{p}_i$, $G_{M_i^n} = G_i$, and (i) X_i^0 is c–one in M_i^0, $P_{M_i^{n+1},c}$ one$_{\phi^n}$–refines $P_{M_i^n,c}$ w.r.t. G_i, (ii) X_{i+I}^0 is d–one in M_i^0, $P_{M_i^{n+1},d}$ one$_{\phi^n}$–refines $P_{M_i^n,d}$ w.r.t. G_i, and, finally, (iii) $P_{M_i^{n+1},f}$ one$_{\psi^n}$–refines $P_{M_i^n,f}$ w.r.t. G_i for all $f \in \mathcal{F}_{\langle M_i \rangle} - \{c, d\}$. Using Lemma A.2, we can show by induction that (i') X_i^n is c–one in M_i^n and (ii') X_{i+I}^n is d–one in M_i^n, while, trivially, (iii') Γ is f–one in M_i^n (for all $n \in \mathbb{N}$, $1 \leq i \leq I$, $f \in \mathcal{F}_{\langle M_i \rangle}$).

We now show that $\langle M_i^{n+1} \rangle$ refines $\langle M_i^n \rangle$ (for all $n \in \mathbb{N}$). We need to show that $G_i(P_{M_i^n,f}, u_{M_i^n}) = G_i(P_{M_i^{n+1},f}, u_{M_i^{n+1}})$ for all $f \in \mathcal{F}_{\langle M_i \rangle}$, $1 \leq i \leq I$, $n \in \mathbb{N}$. We note $u_{M_i^{n+1}}(\phi^n(X_i^n)) = u_{M_i^{n+1}}(X_i^{n+1}) =$

$v_i^i = w_i^i = u_{M_i^n}(X_i^n)$ and similarly for d, while for $f \in \mathcal{F}_{\langle M_i \rangle} - \{c, d\}$ we have (**) $u_{M_i^n} = u_{M_i^{n+1}} \circ \psi^n$ $(1 \leq i \leq I,\ n \in \mathbb{N})$. The claim the follows by applying Lemma A.3 to (i)–(iii) and (i')–(iii'). Finally, we show that (+) $C_{S(\langle M_i^{2n} \rangle)}(\{c, d\}) = \{c\}$ but (++) $C_{S(\langle M_i^{2n+1} \rangle)}(\{c, d\}) = \{d\}$ (for all $n \in \mathbb{N}$). Applying Observation A.1.4 to (i')–(ii'), we find that $X_1^n \cup \ldots \cup X_I^n$ is unanimously c–one in $\langle M_i^n \rangle$ and $X_{I+1}^n \cup \ldots \cup X_{2I}^n$ is unanimously d–one in $\langle M_i^n \rangle$ (for all $n \in \mathbb{N}$). By Definition 2.1, (*) then implies (+) and (++), i.e., c absolutely dominates d in $\langle M_i^{2n} \rangle$ w.r.t. s whereas d absolutely dominates c in $\langle M_i^{2n+1} \rangle$ w.r.t. s (for all $n \in \mathbb{N}$). □

Appendix B Proofs: Robust Aggregation

We assume that all individuals share the same belief type $\mathbf{p}$ and decision rule G. For any decision–theoretic model $M = \langle \mathcal{P}, u, \mathcal{F}, \mathbf{p}, P, G \rangle$, any Γ–partition $\mathcal{P}'$ and any 1–1 mapping $\phi : \mathcal{P} \to \mathcal{P}'$, let the *$\phi$–relabelling* of M be defined as the model $\phi M := \langle \mathcal{P}', u \circ \phi^{-1}, \mathcal{F}, \mathbf{p}, P', G \rangle$ where $P' : \mathcal{F} \to \mathbf{p}(\mathcal{P}')$ is the function such that $P'_f := P_f \circ \bar{\phi}^{-1}$ for all $f \in \mathcal{F}$. For any robust social choice rule S with a wide domain, we have $\langle M_i \rangle \in \mathcal{G}_S$ iff $\langle \phi M_i \rangle \in \mathcal{G}_S$.

Theorem 3.1 and Theorem 3.2.3 *Preparation:* Suppose that S is a robust social choice rule that has a wide domain and that is for models that are one–refinable, cross–product refinable and invariant under relabelling. Suppose S is non–trivial. Hence, there are $\langle M_i^* \rangle, \langle M_i^\dagger \rangle \in \mathcal{G}_S$ such that $\mathcal{F} := \mathcal{F}_{\langle M_i^* \rangle} = \mathcal{F}_{\langle M_i^\dagger \rangle}$, $\langle V_{M_i^*} \rangle = \langle V_{M_i^\dagger} \rangle$, and for some $X \subseteq \mathcal{F}$ and $g \in X$, (+) $g \in C_{S(\langle M_i^* \rangle)}(X)$ but $g \notin C_{S(\langle M_i^\dagger \rangle)}(X)$. Let $N_1 := |\mathcal{P}_{\langle M_i^* \rangle}|$ and $N_2 := |\mathcal{P}_{\langle M_i^\dagger \rangle}|$. Then there is a Γ–partition $\mathcal{P}'$ with $|\mathcal{P}'| = N_1 \cdot N_2$ each of whose elements is at least countably infinite. We enumerate the elements of $\mathcal{P}_{\langle M_i^* \rangle}$, $\mathcal{P}_{\langle M_i^\dagger \rangle}$ and $\mathcal{P}'$ such that $\mathcal{P}_{\langle M_i^* \rangle} = \{C_{m_1} | m_1 = 1, 2, \ldots, N_1\}$, $\mathcal{P}_{\langle M_i^\dagger \rangle} = \{D_{m_2} | m_2 = 1, 2, \ldots, N_2\}$, and $\mathcal{P}' = \{X_{m_1, m_2} | m_1 = 1, 2, \ldots, N_1; m_2 = 1, 2, \ldots, N_2\}$. We define $\mathcal{P}^* := \{\bigcup\{X_{m_1,1}, \ldots, X_{m_1,N_2}\} | m_1 = 1, 2, \ldots, N_1\}$ and $\mathcal{P}^\dagger := \{\bigcup\{X_{1,m_2}, \ldots, X_{N_1,m_2}\} | m_2 = 1, 2, \ldots, N_2\}$. Obviously, $\mathcal{P}^*$ and $\mathcal{P}^\dagger$ are Γ–partitions and subsets of $[\mathcal{P}']$. We define $\phi^* : \mathcal{P}_{\langle M_i^* \rangle} \to \mathcal{P}^*$ by $\phi^*(C_{m_1}) := \bigcup\{X_{m_1,1}, \ldots, X_{m_1,N_2}\}$ and $\phi^\dagger : \mathcal{P}_{\langle M_i^\dagger \rangle} \to \mathcal{P}^\dagger$ by $\phi^\dagger(D_{m_2}) := \bigcup\{X_{1,m_2}, \ldots, X_{N_1,m_2}\}$ (for $m_1 = 1, 2, \ldots N_1$, $m_2 = 1, 2, \ldots N_2$). Obviously, ϕ^* and $\phi^\dagger$ are 1–1. We write $M_i^{**} := \phi^* M_i^*$ for the ϕ^*–relabelling of M_i^* and $M_i^{\dagger\dagger} := \phi^\dagger M_i^\dagger$ for the $\phi^\dagger$–relabelling of $M_i^\dagger$ and, recalling our restrictions on $\mathcal{G}_S$, we set $\mathbf{p} := \mathbf{p}_{\langle M_i^* \rangle} = \mathbf{p}_{\langle M_i^\dagger \rangle}$ and $G := G_{\langle M_i^* \rangle} = G_{\langle M_i^\dagger \rangle}$ $(1 \leq i \leq I)$. Since S has a wide domain, we have $\langle M_i^{**} \rangle, \langle M_i^{\dagger\dagger} \rangle \in \mathcal{G}_S$. We have $V_{M_{kl}^*} = V_{M_{kl}^{**}}$ and $V_{M_{kl}^\dagger} = V_{M_{kl}^{\dagger\dagger}}$ (for all $1 \leq k, l \leq I$) because G is invariant under relabelling. Let S' be an ex ante social choice rule associated with S. Then $S(\langle M_i^{**} \rangle) = S'(\langle M_{kl}^{**} \rangle) = S'(\langle M_{kl}^{\dagger\dagger} \rangle) = S(\langle M_i^{\dagger\dagger} \rangle)$ yields (+') $g \in C_{S(\langle M_i^{**} \rangle)}(X)$ but $g \notin C_{S(\langle M_i^{\dagger\dagger} \rangle)}(X)$ (from (+)). Trivially, $X \cap X' \in \mathcal{P}'$ for every $X \in \mathcal{P}^*$, $X' \in \mathcal{P}^\dagger$. Since $\mathbf{p}$ is cross–product refinable, there thus exists $P'_i : \mathcal{F} \to \mathbf{p}(\mathcal{P}')$ such that, for all $f \in \mathcal{F}$ and all $1 \leq i \leq I$, $P'_{i,f}$ refines $P_{M_i^{**},f}$ as well as $P_{M_i^{\dagger\dagger},f}$. For every $1 \leq i \leq I$, let the functions $v_i, w_i \in \mathbf{u}(\mathcal{P}')$ be defined by $v_i(X_{m_1,m_2}) := u_{M_i^{**}}(\bigcup\{X_{m_1,1}, \ldots, X_{m_1,N_2}\}) = u_{M_i^*}(C_{m_1})$ and $w_i(X_{m_1,m_2}) := u_{M_i^{\dagger\dagger}}(\bigcup\{X_{1,m_2}, \ldots, X_{N_1,m_2}\}) = u_{M_i^\dagger}(D_{m_2})$ for all $m_1 = 1, 2, \ldots N_1$, $m_2 = 1, 2, \ldots N_2$. Since S has a wide domain, there exists $\langle M'_i \rangle \in \mathcal{G}_S$ such that (for all $1 \leq i \leq I$) $\mathcal{P}_{\langle M'_i \rangle} = \mathcal{P}'$, $u_{M'_i} = v_i$, $\mathcal{F} \subseteq \mathcal{F}_{\langle M'_i \rangle}$, $\mathbf{p}_{\langle M'_i \rangle} = \mathbf{p}$, $G_{\langle M'_i \rangle} = G$, and $P_{M'_i,f} = P'_{i,f}$ for all $f \in \mathcal{F}$. By the invariance of G under empty refinements, it follows that (*) $G(P_{M'_k,f}, v_l) = G(P_{M_k^{**},f}, u_{M_l^{**}})$ and $G(P_{M'_k,f}, w_l) = G(P_{M_k^{\dagger\dagger},f}, u_{M_l^{\dagger\dagger}})$ whence $V_{M'_{kl}}(f) = V_{M_{kl}^{**}}(f)$ (for all $f \in \mathcal{F}$, $1 \leq k, l \leq I$). This completes our preparation and we now turn to finishing the proofs for our two separate propositions.

Proof of Theorem 3.1: $\langle M'_i \rangle$ refines both $\langle M_i^{**} \rangle$ and $\langle M_i^{\dagger\dagger} \rangle$ because $V_{M'_i}(f) = V_{M_i^{**}}(f)$ (by (*)) and $V_{M_i^{**}}(f) = V_{M_i^*}(f) = V_{M_i^\dagger}(f) = V_{M_i^{\dagger\dagger}}(f)$ (for all $1 \leq i \leq I$ and $f \in \mathcal{F}$). Now suppose S were stable and non–trivial. This would require that $g \in C_{S(\langle M'_i \rangle)}(X)$ and $g \notin C_{S(\langle M'_i \rangle)}(X)$ (by (+')). Contradiction! Hence, S is not stable if it is non–trivial. □

Proof of Theorem 3.2.3: We recall that every $X_{m_1,m_2} \in \mathcal{P}'$ is at least countably infinite ($m_1 = 1,2,\dots N_1$, $m_2 = 1,2,\dots N_2$). Proceeding as in Theorem 2.2, we can hence enumerate the elements of some countably infinite subset of X_{m_1,m_2} ($m_1 = 1,2,\dots N_1$, $m_2 = 1,2,\dots N_2$) in a sequence $x^1_{m_1,m_2}$, $x^2_{m_1,m_2}, \dots, x^n_{m_1,m_2}, \dots$ (where $x^n_{m_1,m_2} \neq x^{n'}_{m_1,m_2}$ for any $n, n' \in \mathbb{N}^+$). We then define a sequence $\langle \mathcal{P}^n_{m_1,m_2} \rangle$ of partitions of X_{m_1,m_2} by $\mathcal{P}^n_{m_1,m_2} = \{\{x^1_{m_1,m_2}\}, \dots, \{x^n_{m_1,m_2}\}, X_{m_1,m_2} - \{x^1_{m_1,m_2}, \dots, x^n_{m_1,m_2}\}\}$ for all $n \in \mathbb{N}^+$. We write $X^0_{m_1,m_2} := X_{m_1,m_2}$ and $X^n_{m_1,m_2} := X_{m_1,m_2} - \{x^1_{m_1,m_2}, \dots, x^n_{m_1,m_2}\}$ for all $n \in \mathbb{N}^+$. Let $\mathcal{P}^0 := \mathcal{P}' = \{X_{m_1,m_2} | m_1 = 1,2,\dots N_1; m_2 = 1,2,\dots N_2\}$ and $\mathcal{P}^n := \bigcup \{\mathcal{P}^n_{m_1,m_2} | m_1 = 1,2,\dots N_1; m_2 = 1,2,\dots N_2\}$ for all $n \in \mathbb{N}^+$. Let $\phi^0 : \mathcal{P}^0 \to \mathcal{P}^1$ and $\psi^0 : \mathcal{P}^0 \to \mathcal{P}^1$ be defined by $\phi^0(X^0_{m_1,m_2}) := X^1_{m_1,m_2}$ and $\psi^0(X^0_{m_1,m_2}) := \{x^1_{m_1,m_2}\}$ and, for any $n \in \mathbb{N}^+$, let $\phi^n : \mathcal{P}^n \to \mathcal{P}^{n+1}$ and $\psi^n : \mathcal{P}^n \to \mathcal{P}^{n+1}$ be defined by $\phi^n(\{x^{n'}_{m_1,m_2}\}) := \{x^{n'}_{m_1,m_2}\}$ and $\psi^n(\{x^{n'}_{m_1,m_2}\}) := \{x^{n'}_{m_1,m_2}\}$ for all $n' \leq n$ ($n' \in \mathbb{N}^+$) and by $\phi^n(X^n_{m_1,m_2}) := X^{n+1}_{m_1,m_2}$ and $\psi^n(X^n_{m_1,m_2}) := \{x^{n+1}_{m_1,m_2}\}$ (for all $m_1 = 1,2,\dots N_1$, $m_2 = 1,2,\dots N_2$). Define $F^n : \mathcal{P}^0 \to \mathcal{P}^{n+1}$ by $F^n := \phi^n \circ \dots \circ \phi^0$ for all $n \in \mathbb{N}$. Trivially, $\mathcal{P}^{n+1}$ both ϕ^n–refines and ψ^n–refines $\mathcal{P}^n$; moreover, $\mathcal{P}^{n+1}$ F^n–refines $\mathcal{P}^0$ (for all $n \in \mathbb{N}$). We now recursively define a sequence $\langle u^n_1, \dots, u^n_I \rangle_{n \in \mathbb{N}}$ of vectors of individual utility functions such that, for each $n \in \mathbb{N}$ and $1 \leq i \leq I$, $u^n_i \in \mathbf{u}(\mathcal{P}^n)$. For all even $n \in \mathbb{N}$, let $u^n_i(X^n_{m_1,m_2}) := v_i(X_{m_1,m_2})$ and, for all odd $n \in \mathbb{N}$, let $u^n_i(X^n_{m_1,m_2}) := w_i(X_{m_1,m_2})$ (for all $1 \leq i \leq I$, $m_1 = 1,2,\dots N_1$, $m_2 = 1,2,\dots N_2$). Finally, for any $n, n' \in \mathbb{N}^+$ with $n' \leq n$, let $u^{n+1}_i(\{x^{n'}_{m_1,m_2}\}) := u^n_i(\{x^{n'}_{m_1,m_2}\})$ and, for any $n \in \mathbb{N}$, let $u^{n+1}_i(\{x^{n+1}_{m_1,m_2}\}) := u^n_i(X^n_{m_1,m_2})$ (for all $1 \leq i \leq I$, and $1 \leq k \leq 2I$). Notice that, for any $n \in \mathbb{N}$ and $1 \leq i \leq I$, we have not only (**) $u^n_i = u^{n+1}_i \circ \psi^n$, but also (***) $v_i = u^0_i$, $v_i = u^{2n+2}_i \circ F^{2n+1}$, and $w_i = u^{2n+1}_i \circ F^{2n}$. It can be shown by induction that there is a sequence $\langle M^n_1, \dots, M^n_I \rangle_{n \in \mathbb{N}}$ of vectors of individual decision–theoretic models such that $\langle M^0_1, \dots, M^0_I \rangle = \langle M'_1, \dots, M'_I \rangle$ and, for all $n \in \mathbb{N}$, $\langle M^n_1, \dots, M^n_I \rangle \in \mathcal{G}_S$ and, for all $1 \leq i \leq I$, $\mathcal{P}_{\langle M^n_i \rangle} = \mathcal{P}^n$, $u_{M^n_i} = u^n_i$, $\mathcal{F}_{\langle M^n_i \rangle} \subseteq \mathcal{F}_{\langle M^{n+1}_i \rangle}$, $\mathbf{p}_{M^n_i} = \mathbf{p}$, $G_{M^n_i} = G$, (i) for all $f \in \mathcal{F}$, $P_{M^{n+1}_i, f}$ one$_{\phi^n}$–refines $P_{M^n_i, f}$ w.r.t. G, and (ii) for all $f \in \mathcal{F}_{\langle M^n_i \rangle} - \mathcal{F}$, $P_{M^{n+1}_i, f}$ one$_{\psi^n}$–refines $P_{M^n_i, f}$ w.r.t. G. Using the transitivity of belief refinements (p. 3), Lemma A.2 and (i), we can show by induction that $P_{M^n_i, f}$ one$_{F^n}$–refines $P_{M'_i, f}$ w.r.t. G for all $f \in \mathcal{F}$, $1 \leq i \leq I$, $n \in \mathbb{N}^+$. Applying Lemma A.3 to (***), we find that $G(P_{M^{2n}_k, f}, u_{M^{2n}_l}) = G(P_{M'_k, f}, v_l)$, $G(P_{M^{2n+1}_k, f}, u_{M^{2n+1}_l}) = G(P_{M'_k, f}, w_l)$ and, by (*), both $V_{M^{2n}_{kl}}(f) = V_{M^{**}_{kl}}(f)$ and $V_{M^{2n+1}_{kl}}(f) = V_{M^{\dagger\dagger}_{kl}}(f)$ (for all $f \in \mathcal{F}$, $n \in \mathbb{N}$, $1 \leq k, l \leq I$).

We now show that M^{n+1}_i refines M^n_i, for all $n \in \mathbb{N}$ and $1 \leq i \leq I$. We need to show that $V_{M^n_i}(f) = V_{M^{n+1}_i}(f)$ for all $f \in \mathcal{F}_{\langle M^n_i \rangle}$, $1 \leq i \leq I$, $n \in \mathbb{N}$. If $f \in \mathcal{F}$, this follows from the fact that $V_{M^{2n}_i}(f) = V_{M^{**}_i}(f) = V_{M^{\dagger\dagger}_i}(f) = V_{M^{2n+1}_i}(f)$ (for and $1 \leq i \leq I$, $n \in \mathbb{N}$). If $f \in \mathcal{F}_{\langle M^n_i \rangle} - \mathcal{F}$, the claim follows by applying Lemma A.3 to (ii) and (**). Next, if S is IIA and non–trivial, then we have, for any $n \in \mathbb{N}$, $C_{S(\langle M^{2n}_i \rangle)}(Y) = C_{S(\langle M^{**}_i \rangle)}(Y)$ and $C_{S(\langle M^{2n+1}_i \rangle)}(Y) = C_{S(\langle M^{\dagger\dagger}_i \rangle)}(Y)$ (for all $Y \in F$). By (+'), we then have $g \in C_{S(\langle M^{2n}_i \rangle)}(X)$ but $g \notin C_{S(\langle M^{2n+1}_i \rangle)}(X)$ (for all $n \in \mathbb{N}$). □

Theorem 3.2.1 *Proof:* Suppose that S is a robust social choice rule that has a wide domain and is for models that are one–refinable. (Note that, for this part of the theorem, we do not require cross–product refinability and invariance under relabelling.) Suppose furthermore that S is Pareto optimal. By definition of a robust social choice rule, there exists $\langle M^*_i \rangle \in \mathcal{G}_S$. Let $\mathbf{p} := \mathbf{p}_{\langle M^*_i \rangle}$, and $G := G_{\langle M^*_i \rangle}$. Let $p \in \mathbf{p}(\Gamma)$. Since G is non–trivial, there are $u, u' \in \mathbf{u}(\Gamma)$ such that $G(p, u) > G(p, u')$. Define $a := u(\Gamma)$, $b := u'(\Gamma)$ and $\alpha := G(p, u)$, $\beta := G(p, u')$. We note that (*) for any Γ–partition $\mathcal{P}$, any $q \in \mathbf{p}(\mathcal{P})$ that refines p, any $C \in \mathcal{P}$ for which C is q, G–one and any $v_a, v_b \in \mathbf{u}(\mathcal{P})$ with $v_a(C) = a$ and $u_b(C) = b$, we have $G(q, v_a) = \alpha$ and $G(q, v_b) = \beta$ (by Lemma A.3). Next, we partition Γ into two at least countably infinite sets X_k ($k = 1,2$). For each $k = 1,2$, we enumerate the elements of some countably infinite subset of X_k in a sequence $x^1_k, x^2_k, \dots, x^n_k, \dots$ (where $x^n_k \neq x^{n'}_k$ for any $n, n' \in \mathbb{N}^+$). For all for all $n \in \mathbb{N}^+$ and $k = 1,2$, let X^n_k, $\mathcal{P}^n$, $\mathcal{P}^n_k$, ϕ^n and ψ^n be defined as in Theorem 2.2. We now recursively define a sequence $\langle u^n_1, \dots, u^n_I \rangle_{n \in \mathbb{N}^+}$ of vectors of individual utility functions such that, for each $n \in \mathbb{N}^+$ and $1 \leq i \leq I$, $u^n_i \in \mathbf{u}(\mathcal{P}^n)$. Let $u^n_j(C) := a$ for all $C \in \mathcal{P}^n$, $2 \leq j \leq I$, $n \in \mathbb{N}^+$. For all even $n \in \mathbb{N}^+$, let $u^n_1(X^n_1) := a$ and $u^n_1(X^n_2) := b$. For all odd $n \in \mathbb{N}^+$, let $u^n_1(X^n_1) := b$ and $u^n_1(X^n_2) := a$. Finally, let $u^1_1(\{x^1_1\}) := u^1_1(\{x^1_2\}) := a$ and, for any $n, n' \in \mathbb{N}^+$ with $n' \leq n$, let $u^{n+1}_1(\{x^{n'}_k\}) := u^n_1(\{x^{n'}_k\})$ and $u^{n+1}_1(\{x^{n+1}_k\}) := u^n_1(X^n_k)$ (for $k = 1,2$). As in Theorem 2.2, we have (**) $u^n_i = u^{n+1}_i \circ \psi^n$ for any $n \in \mathbb{N}^+$ and $1 \leq i \leq I$.

By assumption, there are $c, d \in \Phi$ with $c \neq d$. It can be shown by induction that there is a sequence $\langle M_1^n, \ldots, M_I^n \rangle_{n \in \mathbb{N}^+}$ of vectors of individual decision–theoretic models such that, for all $n \in \mathbb{N}^+$, $\langle M_1^n, \ldots, M_I^n \rangle \in \mathcal{G}_S$ and, for all $1 \leq i \leq I$ and $2 \leq j \leq I$, $\mathcal{P}_{\langle M_i^n \rangle} = \mathcal{P}^n$, $u_{M_i^n} = u_i^n$, $c, d \in \mathcal{F}_{\langle M_i^n \rangle} \subseteq \mathcal{F}_{\langle M_i^{n+1} \rangle}$, $\mathbf{p}_{\langle M_i^n \rangle} = \mathbf{p}$, $G_{\langle M_i^n \rangle} = G$ and (i) $P_{M_i^0, c}$ refines p, $\{x_1^1\}$ is c–one in M_1^0, X_1^1 is c–one in M_j^0, $P_{M_i^{n+1}, c}$ one$_{\phi^n}$–refines $P_{M_i^n, c}$ w.r.t. G (ii) $P_{M_i^0, d}$ refines p, $\{x_2^1\}$ is d–one in M_1^0, X_2^1 is d–one in M_j^0, $P_{M_i^{n+1}, d}$ one$_{\phi^n}$–refines $P_{M_i^n, d}$ w.r.t. $\mathbf{u}, G$ and, finally, (iii) for all $f \in \mathcal{F}_{\langle M_i^n \rangle} - \{c, d\}$, $P_{M_i^{n+1}, f}$ one$_{\psi^n}$–refines $P_{M_i^n, f}$ w.r.t. G. Using Lemma A.2, we can show by induction that (i') $\{x_1^1\}$ is c–one in M_1^n, X_1^n is c–one in M_j^n, (ii') $\{x_2^1\}$ is d–one in M_1^n, X_2^n is d–one in M_j^n and, trivially, (iii') Γ is f–one in M_i^n (for all $n \in \mathbb{N}^+$, $2 \leq j \leq I$, $1 \leq i \leq I$, $f \in \mathcal{F}_{\langle M_i^n \rangle}$). We now show that $\langle M_i^{n+1} \rangle$ refines $\langle M_i^n \rangle$ for all $n \in \mathbb{N}^+$. We need to show that $G(P_{M_i^n, f}, u_{M_i^n}) = G(P_{M_i^{n+1}, f}, u_{M_i^{n+1}})$ for all $f \in \mathcal{F}_{\langle M_i^n \rangle}$, $1 \leq i \leq I$, $n \in \mathbb{N}^+$. We note that $u_{M_1^n}(\{x_1^1\}) = u_{M_1^{n+1}}(\{x_1^1\}) = u_{M_1^{n+1}}(\phi^n(\{x_1^1\}))$ and $u_{M_j^n}(X_1^n) = u_{M_j^{n+1}}(X_1^n) = u_{M_j^{n+1}}(\phi^n(X_1^n))$ $(1 \leq i \leq I,\ 2 \leq j \leq I,\ n \in \mathbb{N}^+)$ and similarly for d. On the other hand, (**) is relevant in case $f \in \mathcal{F}_{\langle M_i^n \rangle} - \{c, d\}$. The claim now follows by applying Lemma A.3 to (i)–(iii), (i')–(iii'). By (*) and the transitivity of belief refinements (p. 3), we find that (for all $n \in \mathbb{N}^+$, $1 \leq k \leq I$, $2 \leq j \leq I$) $V_{M_{kj}^n}(c) = V_{M_{kj}^n}(d) = \alpha$ and $V_{M_{11}^n}(c) = V_{M_{11}^n}(d) = \alpha$, while $V_{M_{j1}^{2n}}(c) = \alpha$, $V_{M_{j1}^{2n}}(d) = \beta$, $V_{M_{j1}^{2n+1}}(c) = \beta$ and $V_{M_{j1}^{2n+1}}(d) = \alpha$. By assumption $\alpha > \beta$. If S is Pareto optimal, it follows that $C_{S(\langle M_i^{2n} \rangle)}(\{c, d\}) = \{c\}$ but $C_{S(\langle M_i^{2n+1} \rangle)}(\{c, d\}) = \{d\}$ (for all $n \in \mathbb{N}^+$). $\square$

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